IMPRS Bootcamp: Homework - Day 1

GQFI

I. Holographic RG:

(Refer to : Heemskerk and Polchinski, 1010.1264) Coordinate system for this exercise : Poincare coordinates, AdS_{d+1} radius = 1.

$$ds^2 = \frac{dz^2 + d\vec{x}^2}{z^2}.$$

The boundary coordinates are $\vec{x}$ which run from 1 to d.

- Take the following gaussian ansatz for the bulk UV wavefunction,

$$\psi_{UV} = \exp\Bigg(- \frac{1}{2h(\ell)} \int \frac{d^d x}{\ell^d} (\tilde{\phi} + g(\ell))^2 \Bigg).$$

  Using the postulate of holographic RG, ie,

$$\psi_{IR} = \int_{k\delta<1} \mathcal{D}M \, \exp\Bigg(- S_0(M) + \int \frac{d^d x}{\ell^d} \tilde{\phi}\mathcal{O} \Bigg).$$

  1. Derive the effective action S, by doing the necessary gaussian integral, i.e., compute

$$Z = e^{-S} = \int \mathcal{D}\tilde{\phi} \, \psi_{IR}\psi_{UV}.$$

  Note how the quantities $g(\ell)$ and $h(\ell)$ from the ansatz for the ultraviolet Euclidean wavefunction appears as couplings in the effective action. In particular you will generate a double trace coupling!

- We shall now study the renormalization group flows in the couplings of the above derived effective action from the bulk. It will be useful to compute :

  - **2. a)**

$$\frac{\delta S}{\delta \mathcal{O}} = ?$$

  - **2. b)**

$$\partial_\ell S = ?$$

The scalar field radial Hamiltonian density defined on $z = \ell$ is given by,

$$\mathcal{H}(\tilde{\pi}, \tilde{\phi}) = \frac{1}{2\ell}\left(\ell^d \tilde{\pi}^2 + \frac{1}{\ell^d} m^2 \tilde{\phi}^2\right).$$

(Optional problem : Show that this is the case, starting from the free scalar field action in AdS_{d+1})

3. Argue that

$$\mathcal{H}(\tilde{\pi}, \tilde{\phi}) = \mathcal{H}(\frac{i}{\ell^d}\mathcal{O}, -\ell^d \frac{\delta S}{\delta \mathcal{O}})$$

and find the expression for $\mathcal{H}$ in terms of the scalar operator $\mathcal{O}$, use result from problem 2a.

4. Now substitute the result from problem 2b and 3 in the following evolution equation (notice this is just the Hamilton-Jacobi equation or, the radial Schrodinger equation),

$$\partial_\ell S = \mathcal{H}(\frac{i}{\ell^d}\mathcal{O}, -\ell^d \frac{\delta S}{\delta \mathcal{O}}).$$

Compare coefficients of $\mathcal{O}^2$ to derive the equation for $\ell \partial_\ell h$. Congratulations, you have found the β function equation of the coupling h. What are the fixed points? Use the value of the fixed points to infer that $\mathcal{O}$ has scaling dimension,

$$\Delta_\pm = \frac{d}{2} \pm \nu.$$

where, $\nu = \sqrt{\frac{d^2}{4} + m^2}$.

II. AdS Schwarzchild Thermodynamics

The Euclidean Schwarzchild solution of AdS_5 is,

$$ds^2 = f d\tau^2 + \frac{dr^2}{f} + r^2 d\Omega_3^2.$$

where,

$$f = 1 + \frac{r^2}{l^2} - \frac{\mu}{r^2}.$$

1. The horizon is the outermost solution of the equation $f(r_+) = 0$. Solve this equation, find the location of the horizon.

2. Focus on the near-horizon geometry and from it find the black hole inverse temperature, β. Note that for a given β there are two possible black hole solutions. The one with lower free energy will be stabler.

Now using the Euclidean on-shell gravity action, I_E we shall find the free energy, $F = -\frac{1}{\beta}\log Z$, where $Z = e^{-I_E}$. On a five dimensional asymptotically AdS spacetime with boundary, I_E takes the form,

$$I_E[g] = -\frac{1}{16\pi G_N}\int_{\mathcal{M}} \sqrt{g}\left(R + \frac{12}{l^2}\right) + \frac{1}{8\pi G_N}\int_{\partial\mathcal{M}} \sqrt{\gamma}K + S_{c.t.}.$$

The first term in the Einstein-Hilbert term, second one is the Gibbons-Hawking term and the last term is a counterterm that need to be added to remove divergences. The boundary of the metric

is at $r = \infty$, which we take to be $r = 1/\epsilon$ and then take $\epsilon \to 0$ while regularizing. We shall now calculate each of these terms. **Hint :** The Ricci scalar for the above metric will turn out to be $-20/l^2$.

3. Find the bulk term and the Gibbons-Hawking terms, expressed in term of β.

4. The above terms have $1/\epsilon^4$ and $1/\epsilon^2$ divergences. To fix this the only choice that makes everything finite is,

$$S_{c.t.} = \frac{3}{8\pi G_N l} \int_{r=1/\epsilon} d^4x \, \sqrt{\gamma} \left(1 + \frac{l^2}{12} R_\gamma \right).$$

Find the finite regularized on-shell action and hence the partition function as a function of temperature. Proceed to compute the energy and the entropy.